THE PARAMETRIC THEORY OF SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

JAMES DOW CAMPBELL

Univ. Urbana

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BY

JAMES DOW CAMPBELL

B. S., Vanderbilt University, 1933

M. A., University of Illinois, 1934

THESIS

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I. Introduction.

The problem considered in this thesis is that of obtaining a fundamental solution for the linear homogeneous parabolic differential equation:

$$(1.1)\qquad L(s) \equiv \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 s}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i \frac{\partial s}{\partial x_i} + a s - \frac{\partial s}{\partial y} = 0$$

where

$$a_{ij}(x_1, \ldots, x_m; y) = a_{ij}(x, y)$$

$$(1.1a)\qquad a_i(x, y; \lambda) = \lambda^H \alpha_i(x, y; \lambda) \sim \lambda^H \alpha_i^*(x, y; \lambda) = \lambda^H \sum_{\nu=0}^{\infty} \alpha_{i,\nu}(x, y) \lambda^{-\nu}$$

$$a(x, y; \lambda) = \lambda^{2H} \alpha(x, y; \lambda) \sim \lambda^{2H} \alpha^*(x, y; \lambda) = \lambda^H \sum_{\nu=0}^{\infty} \alpha_{i,\nu}(x, y) \lambda^{-\nu}$$

H is a positive integer; at least one of $\alpha_{i,0}$ $(i = 1, \ldots, n)$ and α_0 is not zero; $\sim$ means asymptotic.

The problem is an application of a method developed by Trjitzinsky.[1] He showed how a formal solution could be obtained and under what circumstances one could obtain an actual solution. This thesis gives one method by which such an actual solution can be constructed.

The following point sets will be used:

(a) R is a bounded open region of the $(x_1, \ldots, x_m)$ space with closure $\bar{R}$;

(b) $\bar{I}$ is the closed interval $(\eta \leq y \leq y')$;

(c) S is an open simply connected region extending to infinity in the complex λ plane.

It will be assumed that:

(A) the functions a_{ij}, a_i, a $(i,j=1,\dots,n)$ and their partial derivatives with respect to $x_i (i=1,\dots,n)$ and y of as high order as is necessary are continuous for x in $\bar{R}$. y in $\bar{I}$ and λ in S;

(B) for the values of x and y given in (A) the function

$$(1.1b)\quad \sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\xi_j$$

is a positive definite form in ξ_i $(i=1,\dots,n)$

II. The Formal Solution.

In equation (1.1) replace the $a_i (i=1,\dots,n)$ and a by the formal expressions (1.1a). Then the equation (1.1) goes over into a formal equation:

$$(2.1)\quad L^*(S) \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 S}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \lambda^H \alpha_i^* \frac{\partial S}{\partial x_i} + \lambda^{2H}\alpha^* S - \frac{\partial S}{\partial y} = 0$$

This equation is satisfied by a formal series

$$(2.2)\quad S = e^{Q(x,y;\lambda)}\,\sigma(x,y;\lambda)$$

where

$$(2.3)\quad Q(x,y;\lambda) = \sum_{i=0}^{H-1} S_i(x,y)\,\lambda^{H-i}\,;\quad \sigma(x,y;\lambda) = \sum_{\nu=0}^{\infty} \sigma_\nu(x,y)\,\lambda^{-\nu}.$$

The equation for $S_0(x,y)$ is

$$(2.4)\quad e_0(x,y;S_0) \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial S_0}{\partial x_i}\frac{\partial S_0}{\partial x_j} + \sum_{i=1}^{n} \alpha_{i,0}\frac{\partial S_0}{\partial x_i} + \alpha_0 = 0$$

This is an equation of the first order and is of degree two. By the assumption (B) it is actually of degree two. This equation

$e_0(x,y;s_0) = 0$ is called the characteristic equation of (2.1). It has a solution $S_0(x,y)$. If this solution has a zero in the region R and $\bar{I}$ then the formal solution is in fractional powers of λ. This case will be excluded here.

Definition 2.2: Let $S_0(x,y)$ be a continuous solution with continuous first partial derivatives of $e_0(x,y;s_0) = 0$ It will be said that $S_0(x,y)$ is non-singular in R and $\bar{I}$ if for no point of R and $\bar{I}$ do the functions $\frac{\partial e_0(x,y;s_0)}{\partial p_{0k}}$ $\left(p_{0k} = \frac{\partial s_0}{\partial x_k};\ k = 1,\dots,n\right)$ all vanish.

Theorem 2.1: Let the coefficients of the various powers of λ in the formal equation (2.1) be indefinitely differentiable for x in R and y in $\bar{I}$. Let $S_0(x,y)$ be a solution of the characteristic equation $e_0(x,y;s_0) = 0$ which is non-singular in R and $\bar{I}$. Then there exist formal solutions of (2.1) of the form (2.2).

In obtaining this solution the $S_j(x,y)$ $(j = 0,1,\dots,H-1)$ are determined from

$$(2.5)\quad \sum_{\nu=1}^{n} \frac{\partial e_0(x,y;s_0)}{\partial p_{0,\nu}} p_{j\nu} + e_j(x,y;s_0,\dots,s_{j-1}) = 0 \quad \left(j = 1,\dots,H-1;\ p_{j,\nu} = \frac{\partial s_j}{\partial x_\nu}\right)$$

where

$$(2.5a)\quad e_{K+1}^{(K+1)}(x,y;s_0,\dots,s_K) = \sum_{i,j=1}^{n} a_{ij} \sum_{K_1,K_2}^{(K+1)} p_{K_1,i}\, p_{K_2,j} + \sum_{i=1}^{n} p_{K+1,i} \sum_{l=1}^{K+1} a_{i,l} + a_{K+1}$$

and where $\sum_{K_1,K_2}^{(K+1)} p_{K_1,i}\, p_{K_2,j}$ omits the terms with K_1 or $K_2 = K+1$

The $\sigma_0, \sigma_1, \dots$ are determined consecutively from the solutions of

$$(2.6)\quad \delta_k \equiv \sum_{i,j=1}^{n} a_{ij}\sigma_k^{ij} + \sum_{i=1}^{n}\sum_{\ell=0}^{k} \sigma_{k-\ell}^{i}\alpha_{i\ell} + \sum(\sigma_{k-\ell}\alpha_\ell + \sigma_{k-1}^{n+1}p_\ell) = 0$$

III. A Transformation.

For x in a closed subset $\bar{R}_1$ of R, for y in a closed subset $\bar{I}_1$ of $\bar{I}$ and for λ in S,

$$(3.1)\quad \alpha_i(x,y;\lambda) = \sum_{\nu=0}^{q-1}\alpha_{i,\nu}(x,y)\lambda^{-\nu} + R_{iq}(x,y;\lambda)\lambda^{-q} \quad (i=1,\dots,n)$$

$$(3.2)\quad \alpha(x,y;\lambda) = \sum_{\nu=0}^{q-1}\alpha_\nu(x,y)\lambda^{-\nu} + R_q(x,y;\lambda)\lambda^{-q}.$$

q is any fixed integer and

$$|R_{iq}(x,y;\lambda)|\ ,\ |R_q(x,y;\lambda)| \leqq B_q(\bar{R}_1,\bar{I}_1) < \infty\ .$$

The asymptotic relationships are assumed to be differentiable a suitable number of times.

Equation (2.1) is transformed by

$$(3.3)\quad S = e^{Q(x,y;\lambda)}\left[\sigma_t(x,y;\lambda) + \lambda^{-t}z_t(x,y;\lambda)\right]$$

where

$$(3.4)\quad \sigma_t(x,y;\lambda) = \sum_{\nu=0}^{t-1}\sigma_\nu(x,y)\lambda^{-\nu}.$$

t is a positive integer such that

$$(3.5)\quad t \leqq q.$$

This leads to an equation

$$(3.6)\quad (L_1 + L_2) + L_3 = 0$$

where,

$$(3.7)\quad L_1 + L_2 = \lambda^{2n-t}e^{Q(x,y;\lambda)}f_t(x,y;\lambda)\ ;\ |f_t(x,y;\lambda)| \leqq C_t(\bar{R}_1,\bar{I}_1) < \infty$$

for x in $\bar{R}_1$, y in $\bar{I}_1$ and λ in S; and

$$(3.8)\quad L_3 = \lambda^{2H-t} e^{Q(x,y;\lambda)} \lambda^t \left\{ \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 z_t}{\partial x_i \partial x_j} + \sum_{i=1}^{m} \lambda^H \frac{\partial z_t}{\partial x_i} \left[\alpha_i + 2 \sum_{j=1}^{m} a_{ij} \sum_{\nu=0}^{H-1} b_{\nu j} \lambda^{-\nu} \right] + \lambda^H \frac{\partial z_t}{\partial y}(-\lambda^H) + \lambda^{2H} z_t \left[\alpha + \lambda^{-H} \sum_{i,j=1}^{m} a_{ij} \sum_{\nu=0}^{H-1} b_{\nu i} \lambda^{-\nu} - \lambda^{-H} \sum_{\nu=0}^{H-1} b_{\nu,n+1} \lambda^{-\nu} + \sum_{i,j}^{m} a_{ij} \sum_{\nu=0}^{H-1} b_{\nu i} \lambda^{-\nu} \sum_{\nu_1=0}^{H-1} b_{\nu_1 j} \lambda^{-\nu_1} + \sum_{i=1}^{m} \sum_{\nu=0}^{\infty} \alpha_{i\nu} \lambda^{-\nu} \sum_{\nu_1=0}^{H-1} b_{\nu_1 i} \lambda^{-\nu_1} \right] \right\}$$

The key to the asymptotic theory of the homogeneous differential equation (1.1) lies in the possibility of obtaining a solution z_t of (3.6) such that

$$(3.9)\quad |z_t(x,y;\lambda)| \leq |\lambda^{\wedge}| \cdot \mathfrak{z}_t(\bar{R}_1, \bar{I}_1) \; ; \; \mathfrak{z}_t(\bar{R}_1, \bar{I}_1) < \infty$$

for x in $\bar{R}_1$, y in $\bar{I}_1$ and λ in S. $\mathfrak{z}_t(\bar{R}_1, \bar{I}_1)$ is independent of λ. $\wedge$ is a number such that $0 < \wedge < t$. Moreover it is desirable to have

$$(3.10)\quad \lim_{t \to \infty} (t - \wedge) = \infty$$

whenever the nature of the problems permits to take t arbitrarily great.

Finally write (3.6) in the form

$$(3.11)\quad L(z_t) = \lambda^{2H} f_t(x,y;\lambda)$$

Here $L(z_t)$ is the expression inside parenthesis in (3.8).

IV. The Solution of Equation (3.11).

Dressel[2] has solved a case similar to (3.11) but not

containing a parameter . Following his method replace $z_t(x,y;\lambda)$ by

$$(4.1)\quad \Gamma(x,y;\xi,\eta) = z(x,y;\xi,\eta) + \int_{\eta}^{y}\int_{\bar{R}_1} z(x,y;s,\tau) f(s,\tau;\xi,\eta)\, ds\, d\tau$$

(x,ξ) are in $\bar{R}_1$, (y,η) are in $\bar{I}_1$, $z(x,y;\xi,\eta)$ is a known function,(2) $f(s,\tau;\xi,\eta)$ is an unknown function satisfying certain properties.(2) With $z_t(x,y;\lambda)$ replaced by $\Gamma(x,y;\xi,\eta)$ equation (3.11) becomes:

$$(4.2)\quad L(z) - f(x,y;\xi,\eta) + \int_{\eta}^{y}\int_{\bar{R}_1} L[z(x,y;s,\tau)] f(s,\tau;\xi,\eta)\, ds\, d\tau = \lambda^{2H} f_t(x,y;\lambda)$$

This is an integral equation for the unknown function $f(s,\tau;\xi,\eta)$

Define the new operation:

$$(4.3)\quad L_\lambda^q(z) \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + \lambda^H\left\{\sum_{i=1}^{n}\frac{\partial z}{\partial x_i}\beta_i + (-\lambda^{-H})\frac{\partial z}{\partial y}\right\} + \lambda^{2H} z\beta$$

where

$$(4.3a)\quad \beta_i = \sum_{\nu=0}^{q-1}\beta_{i\nu}(x,y)\lambda^{-\nu};\ \beta_{i,0} = 2\sum_{j=1}^{n} a_{ij} b_{0j};\ \beta_{i\nu} = \alpha_{i\nu} + 2\sum_{j=1}^{n} a_{ij} b_{\nu j}\ (i=1,\dots,n)$$

$$(4.3b)\quad \beta = \sum_{\nu=0}^{q-1}\beta_\nu(x,y)\,\lambda^{-\nu}$$ i.e. it is the terms in the coefficient of z_t in (3.8) containing $\lambda^{-\nu}$ with $\nu \le q-1$. The operator $L_\lambda^q(z)$ satisfies the following conditions:

$$(4.4a)\quad |L_\lambda^q(z)| \le O(|\lambda|^{2H})$$

$$(4.4b)\quad |L_\lambda^q(z) - \lambda^{2H} f_t(x,y;\lambda)| \le O(|\lambda|^{2H})$$

$$(4.4c)\quad L(z) = L_\lambda^q(z) + \lambda^{2H-q} G(x,y;\xi,\eta)$$

where

$|G(x,y;\xi,\eta)| \leq G_0$ = a constant for (x,ξ) in $\bar{R}_1$; (y,η) in $\bar{I}_1$; λ in S and $|\lambda| \geq \lambda_0$.

Replacing $L(z)$ in (4.2) by (4.4c) the result is:

$$(4.5)\quad L_\lambda^q(z) + \lambda^{2H-q}G - f + \int_\eta^y \int_{\bar{R}_1} L(\tfrac{q}{\lambda}[z(x,y;s,\tau)]\, f(s,\tau;\xi,\eta)\, ds\, d\tau + \lambda^{2H-q}\int_\eta^y \int_{\bar{R}_1} G(x,y;s,\tau;\lambda)\, f(s,\tau;\xi,\eta)\, ds\, d\tau = \lambda^{2H} f_t(x,y;\lambda) .$$

This may be abbreviated to:

$$(4.6)\quad \Gamma_q(f) - \lambda^{2H-q}\int_\eta^y \int_{\bar{R}_1} G f\, ds\, d\tau = L_q^\lambda(z) + \lambda^{2H-q}G - \lambda^{2H} f_t$$

where,

$$(4.6a)\quad \Gamma_q(f) \equiv f - \int_\eta^y \int_{\bar{R}_1} L_\lambda^q(z) f\, ds\, d\tau .$$

Define the set of integral equations

$$(4.7)\quad \begin{aligned} \Gamma_q(f_0) &= L(x,y;\xi,\eta;\lambda) \\ \Gamma_q(f_K) &= L(x,y;\xi,\eta;\lambda) + \lambda^{2H-q}\int_\eta^y \int_{\bar{R}_1} G(x,y;s,\tau;\lambda) f_{K-1}(s,\tau;\xi,\eta)\, ds\, d\tau \\ &(K = 1, 2, \dots) . \end{aligned}$$

Let the resolvent kernel of this set of integral equations be

$$(4.8)\quad \Lambda(x,y;s,\tau;\lambda) = \sum_{\nu=0}^{\infty} K^{(\nu)}(x,y;s,\tau;\lambda)$$

where

$$(4.8a)\quad K^0 = L_\lambda^q[z(x,y;s,\tau)] \; ; \; K^{(\nu)} = \int_\eta^y \int_{\bar{R}_1} K^{(0)}(x,y;s,\tau) K^{(\nu-1)}(s,\tau;\xi,\eta)\, ds\, d\tau \; ; \; (\nu = 1,2,\dots)$$

It is assumed that $\Lambda(x,y;s,\tau;\lambda)$ is a continuous function for

(x,s) in $\bar{R}_1$, (y,τ) in $\bar{I}$, and λ in S. Then Λ also satisfies $|\Lambda| < \Lambda_0 =$ constant in this region.

Replace the set of equations (4.7) by a set defined in the following way. Put

$$(4.9)\qquad \omega_0 = f_0$$
$$\omega_k = f_k - f_{k-1} \qquad (k = 1, 2, \ldots)$$

Then this set of functions ω_k $(k = 0, 1, 2, \ldots)$ satisfies the set of integral equations

$$(4.10)\qquad \Gamma_q(\omega_0) = L$$
$$\Gamma_q(\omega_k) = \lambda^{2H-q} \int_\eta^y \int_{\bar{R}_1} G\,\omega_{k-1}\, ds\, d\tau \; ; \quad (k = 1, 2, \ldots)$$

The solution for ω_k is given by

$$(4.11)\qquad \omega_k = \lambda^{-\sigma} \int_\eta^y \int_{\bar{R}_1} G\,\omega_{k-1}\, ds\, d\tau + \lambda^{-\sigma} \int_\eta^y \int_{\bar{R}_1} \Lambda\, ds\, d\tau \int_\eta^\tau \int_{\bar{R}_1} G\,\omega_{k-1}\, ds_1\, d\tau_1$$

where $\sigma = q - 2H > 0$ because q can be an arbitrary integer. Furthermore ω_k satisfies the inequality

$$(4.12)\qquad |\omega_k| < |\lambda|^{2H-\sigma k} L_0 G_0^k V^k (1 + \Lambda_0 V)^{k+1} \; ; \quad (k = 1, 2, \ldots)$$

where L_0, G_0 and V are constants.

Now

$$(4.13)\qquad \int(x,y;\xi,\eta;\lambda) = f_0 + \sum_{k=1}^{\infty} (f_k - f_{k-1}) = \omega_0 + \sum_{k=1}^{\infty} \omega_k = \lim_{k=\infty} f_k = f$$

Here f is a continuous solution of (4.2); for the series $\int$ is a serious of continuous functions and it is absolutely and uniformly convergent if $|\lambda| > [G_0 V (1 + \Lambda_0 V)]^{\frac{1}{\sigma}}$. Note that by (4.1) this determines $\Gamma(x,y;\xi,\eta)$ and so determines Z_f. But a

bounded solution for z_t gives an actual solution of equation (1.1). This solution is, by (3.3),

$$(4.14) \qquad S = e^{Q(x,y;\lambda)}\left[\sigma_t(x,y;\lambda) + \lambda^{-t} z_t(x,y;\lambda)\right]$$

Theorem 4.1: Let the equation

$$(1.1) \qquad L(s) \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 s}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial s}{\partial x_i} + as - \frac{\partial s}{\partial y} = 0$$

where the coefficients are specified in (1.1a) satisfy the conditions of Theorem 2.1 (page 3). If, in addition, the following conditions are satisfied:

(1) (A) (see page 2);

(2) (B) (see page 2);

(3) $|R_{iq}(x,y;\lambda)|$, $|R_q(x,y;\lambda)| \leq B_q(\bar{R}_1, \bar{I}_1) < \infty$, $(i = 1, 2, \ldots, n)$ (x in $\bar{R}_1$, y in $\bar{I}_1$ and λ in S, q is an arbitrary positive integer); the asymptotic relationships (3.1) and (3.2) are assumed to be differentiable with respect to $x_i (i = 1, \ldots, n)$ and y a suitable number of times;

(4) $\Delta L(x,y;s,\tau;\lambda)$ (see (4.8) and (4.8a)) is a continuous function for (x,s) in $\bar{R}_1$, (y,τ) in $\bar{I}_1$ and λ in S;

then equation (1.1) has the properties:

(1) It has a solution $\bar{S} = e^{Q(x,y;\lambda)}\left[\sigma_t(x,y;\lambda) + \lambda^{-t} z_t(x,y;\lambda)\right]$

(2) The solution $\bar{S} \underset{(t)}{\sim} S$; where S is the formal solution of (2.11).

The results (1) and (2) hold for x in $\bar{R}_1$, y in $\bar{I}_1$, and λ in S.

References.

(1) W. J. Trjitzinsky, Analytic theory of parametric linear partial differential equations. Récueil Mathematique, T.15(57), N. 2.

(2) F. G. Dressel, The Fundamental Solution of the Parabolic Equation. Duke Mathematical Journal, Vol.7, December 1940.

(3) Vito Volterra, Théorié Générale Des Fonctionnelles, Paris, 1936.

VITA

James Dow Campbell was born in Nashville, Tennessee on October 26, 1907. He attended the public elementary and high schools of that city. In 1929 he entered Vanderbilt University and was graduated cum laude in 1933. He entered the Graduate School of the University of Illinois in September 1933, and received the degree of Master of Arts in Mathematics in 1934. From 1934 until February 1938 he was a graduate assistant in the Department of Mathematics at the University of Illinois. From February 1938 until October 1945 he was an instructor in Mathematics at Rensselaer Polytechnic Institute. In October 1945 he was granted a leave of absence from Rensselaer Polytechnic Institute to complete graduate work in Mathematics at the University of Illinois.

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